

Cooperation in the noisy case: Prisoner's dilemma game on two types of regular random graphs

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We have studied an evolutionary prisoner's dilemma game with players located on two types of random regular graphs with a degree of 4. The analysis is focused on the effects of payoffs and noise (temperature) on the maintenance of cooperation. When varying the noise level and/or the highest payoff, the system exhibits a second-order phase transition from a mixed state of cooperators and defectors to an absorbing state where only defectors remain alive. For the random regular graph (and Bethe lattice) the behavior of the system is similar to those found previously on the square lattice with nearest neighbor interactions, although the measure of cooperation is enhanced by the absence of loops in the connectivity structure. For low noise the optimal connectivity structure is built up from randomly connected triangles.

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One of the central questions in evolutionary game theory is to find necessary conditions and mechanisms that result in cooperation among selfish individuals. Naturally, several mechanisms have already been explored such as kin selection [1], retaliating behavior [2], voluntary participation [3], or development of reputation [4].

A widely studied toy model, which illustrates the conflict between cooperation and selfish behavior, is the prisoner's dilemma game (PDG). In the (two-player and one-shot) PDG [5,6], the players simultaneously decide whether to cooperate or defect in order to maximize their individual payoff. The dilemma is based on the fact that defection brings higher income independently of the other player's decision. But if both players defect they receive significantly lower payoff than in the case of mutual cooperation.

The introduction of short-range interaction between the spatially distributed players can explain the formation of cooperation for iterated evolutionary PDGs [7,8], even for the case when players can follow one of the two simplest strategies "always cooperate" (*C*) and "always defect" (*D*). In multiagent evolutionary PDGs, the players gain their income from games with their neighbors. According to the Darwinian selection principle, the less successful strategy is replaced by a more successful strategy adopted from their neighborhood [9,10].

Several studies followed the pioneering work of Nowak *et al.* [7,8] in the field of evolutionary PDGs analyzing many types of connectivity structures (lattices [11–13], diluted lattices [8], social networks [14–17], hierarchical graphs [18], scale-free networks [19], preferential selection of a neighbor [20]) to clarify the possible role of the topology. In Ref. [21], self-organizing networks controlled by proper dynamics were applied to find the most advantageous structure for cooperation. Our previous study [22] reveals the importance of clique percolation [23,24].

In the present Brief Report, as a continuation of our previous work [22], we explore the effect of noise parameter on the PDG by extending the connectivity structures to random graphs. Our observations suggest that the effects of noise (temperature) on the stationary concentration of cooperators are greatly affected by the underlying structure.

To investigate the influence of the noise and connectivity structures, we have considered an evolutionary PDG with players located on the sites of a regular graph with a connectivity of four ($z=4$). The restriction to regular random graph serves to make the comparison easier and to avoid undesired effects due to the various degrees of nodes (different sizes of neighborhoods) [8,19,25,26].

The players can follow one of the above mentioned two strategies whose distribution is described by the formalism of the two-state Potts model: the possible state of the site x is $s_x=C$ or D (cooperator or defector). The strategy adoption mechanism [22] is based on the rescaled version of the payoff matrix introduced by Nowak *et al.* [7]:

$$\mathbf{A} = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad 1 < b < 2. \quad (1)$$

During the evolutionary process the randomly chosen player at site x can adopt the strategy of one of the (randomly chosen) coplayers (at site y) with a probability depending on the payoff difference ($U_x - U_y$),

$$W[s_x \leftarrow s_y] = \frac{1}{1 + \exp[(U_x - U_y)/K]}, \quad (2)$$

where K characterizes the magnitude of noise involving many different effects (fluctuations in payoffs, errors in decision, individual trials, etc.) [13,27].

The present analysis is focused on two connectivity structures displayed in Fig. 1. Henceforth, we will refer to the

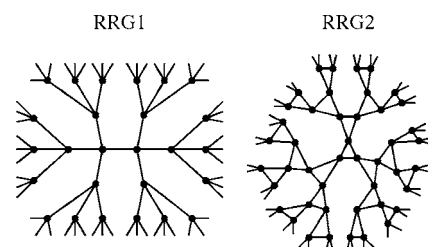


FIG. 1. Two types of random regular graphs (RRGs) on which an evolutionary prisoner's dilemma game is studied.

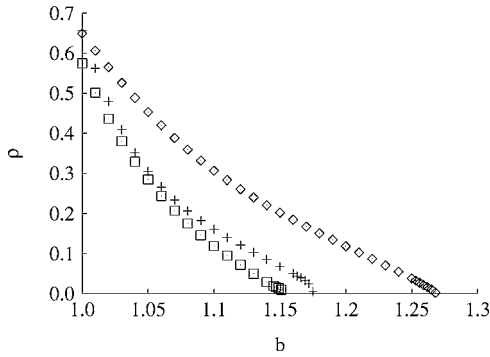


FIG. 2. Monte Carlo results for the concentration of cooperators vs b for three different noise values: $K=0.1$ (pluses), 0.3 (diamonds), and 1.2 (squares) on RRG1.

structures as RRG1 and RRG2, respectively. The RRG1 random regular graph is the simulated version of the Bethe lattice, because RRG1 is locally similar to a tree in the large-size limit (when the number of sites goes to infinity). For a finite number N of sites, however, RRG1 has loops. This influence seems to be negligible when choosing sufficiently large systems. To avoid confusion and for better visualization, Fig. 1 represents an “ideal” (free of loops) part of RRG1.

RRG2 is a random regular graph of triangles with three neighboring triangles. The triangles of RRG2 are the nodes of the underlying random regular graph (with $z=3$), and between these nodes, the bonds are the common sites of the overlapping triangles. Locally, it resembles to the Kagomé lattice, but—as will be seen later—it has some characteristics of a random regular graph, too. Despite the similar coordination number ($z=4$) the topology of the two random networks differs significantly. In the limit $N \rightarrow \infty$, the concentration of triangles (three-site cliques) vanishes in RRG1, i.e., this structure has a clustering coefficient $C=0$, whereas the overlapping triangles percolate on RRG2, and the clustering coefficient is $C=1/3$.

Since the classical mean-field theory is insensible to the topology, the equation of motion for the concentration of cooperators is identical with the equation of the model on lattices presented in our previous work [22]. The solution of the differential equation suggests that cooperators die out and defection is the only successful strategy for arbitrary values of K and $b > 1$.

As was mentioned in our previous paper [22] the necessary condition for cooperators to survive is the possibility to form clusters where cooperators can assist each other. This may happen on lattices in higher dimensions at certain range of parameter b . The present paper surveys the area of the b - K parameter plane where cooperation can survive.

Figure 2 shows the stationary concentration ρ of cooperators on RRG1 as a function of b for different temperature values. This plot indicates that the variation of ρ on RRG1 is qualitatively similar to those found on the square lattice [22]. These numerical data are obtained from Monte Carlo (MC) simulations performed on large systems (the size varied from $N=4 \times 10^4$ to 4×10^6), and the stationary values are determined by averaging over a sampling time t_s (varied from t_s

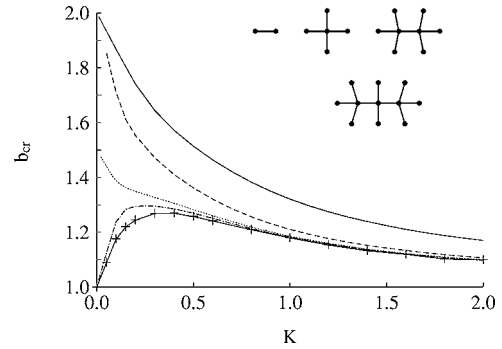


FIG. 3. Critical value of b vs K on RRG1. Symbols come from Monte Carlo simulations, the solid, dashed, dotted, and dash-dotted lines represent the predictions of the dynamical cluster approximation for the 2-, 5-, 8-, and 11-site clusters shown at the top.

$= 10^4$ to 10^6 Monte Carlo steps per sites). The larger N and t_s are used in the close vicinity of the extinction of cooperators because the variance diverges approximately as $\chi \propto \rho^{-2}$ for small concentration and low K . Further difficulties were caused by the size dependence at low K values. The reason for this phenomenon can be related to the existence of small loops whose effect will be discussed later on.

The transition from a fluctuating ($C+D$) phase into the absorbing state frequently belongs to the directed percolation universality class. The critical exponent characterizing the power law decrease of concentration was reproduced when the evolutionary PDG was simulated on different two-dimensional lattices [13,22]. As expected, the change of host lattice to a random graph results in a mean-field type of behavior in $\rho(K)$ as demonstrated in Fig. 2. The stationary concentration of cooperators is independent of the initial state and decreases linearly with $b_{cr} - b$. As b exceeds a threshold value (b_{cr}), cooperation cannot be maintained and the evolution always ends in the homogeneous D state.

We have determined the critical values of b for different noise levels and the results are summarized in Fig. 3. The $b_{cr}(K)$ curve has a maximum at $K \approx 0.37$, and goes to 1 if K goes either to zero or to infinity. This figure can be considered as a phase diagram because the solid line connecting the Monte Carlo data separates the area where cooperators and defectors can coexist from the pure defector area.

The possibility of a similar stochastic resonance was reported by Traulsen *et al.* [28], who considered another evolutionary rule based on the application of the “win-stay-lose-shift” strategies. The appearance of stochastic resonance was directly demonstrated on the square lattice by Perc in a similar PDG by adding random perturbations to the payoffs [29]. A similar type of noise-induced phenomenon, called coherence resonance, has also been observed for excitable dynamical systems [30,31].

The analytical reproduction of this behavior proved to be a very time-consuming task. In contrast to the Monte Carlo results, as detailed in the previous paper [22], the traditional mean-field approximation suggests a sudden change between the homogeneous D and C states at $b_{cr}^{(mf)}(K)=1$. The pair approximation is capable of describing the coexistence of the C and D strategies but it gives a rough estimation for critical

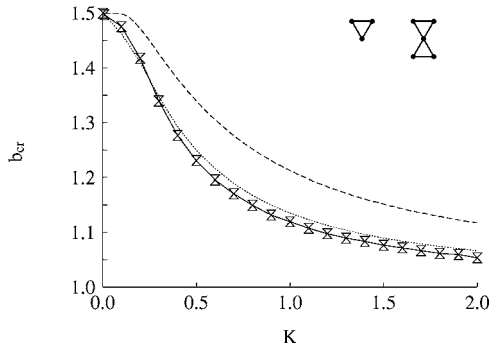


FIG. 4. Phase diagram on RRG2. Symbols denote the MC data. The dashed and dotted lines illustrate the phase boundary between the D and $(C+D)$ phases predicted by the three- and five-site approximations on the clusters shown at the top.

b values, especially in the zero-temperature limit. To eliminate these discrepancies we have to extend the dynamical cluster approximations (considered as generalized mean-field methods) on the Bethe lattice [32]. When applying these techniques we derive equations of motion for all possible configurational probabilities on large clusters and search numerically for the stationary concentrations by integrating the equations of motion with respect to time (further details on these methods will be given elsewhere). Figure 3 shows that both the five- and eight-site approximations predict incorrect results in the zero-temperature limit although their predictions become more and more accurate in the high-temperature region. Besides, one can observe relevant improvement when comparing the results of the five- and eight-site approximations at the limit $K \rightarrow 0$. This fact inspired us to extend this method to the level of 11-site approximations. As shown in Fig. 3 this level is already capable of describing the disappearance of cooperation as K goes to 0. The above series of results emphasize the importance of long-range correlations and/or the absence of loops in the connectivity structures for the limit $K \rightarrow 0$.

Basically different behavior is found on the RRG2 structure as demonstrated in Fig. 4. It is conjectured previously that the function $b_{cr}(K)$ decreases monotonically to 1 if K is increased for those connectivity structures where the overlapping triangles span the whole system. The present data support this conjecture.

In Fig. 4 we compare the MC data with the predictions of three- and five-site approximations. It is worth mentioning that qualitatively similar results are obtained when the connectivity structure is given by the Kagomé lattice [22]. The RRG2 and Kagomé lattices are locally similar: the overlapping triangles have only one common site. This is the main reason why the three- and five-site approximations predict the same results on the RRG2 and Kagomé lattices. In fact, the structure of RRG2 fits well to the conditions of the five-site approximations, overestimating both ρ and b_{cr} for the Kagomé lattice. At the same time, the results of this approach agree very well with the MC data on RRG2 (see Fig. 4). We have to emphasize that this is the connectivity structure providing the highest measure of cooperation among regular structures (if $z=4$) for low K values (for a comparison see Fig. 5). At the first glance it is a surprising result

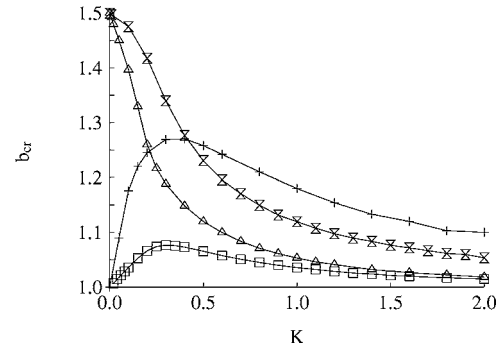


FIG. 5. Phase boundaries between the D and $(C+D)$ phases on RRG1 (pluses), RRG2 (double-triangle symbols), square lattice (squares), and Kagomé lattice (triangles).

because the introduction of spatial connectivity structures was motivated by the possibility of the formation of C colonies [2,7].

The above results have helped us to deduce a simple explanation justifying the importance of the one-site overlapping triangles in the connectivity structures. Let us assume that one of the triangles is occupied by cooperators in the sea of defectors. Within this triangle, the cooperator's income is 2, the neighboring defectors receive b , and all the other D 's get nothing. In this situation the most probable evolutionary process is that one of the neighboring defectors adopts the strategy of the more successful cooperators (for low K). The state of this new cooperator is not stable and it can be switched into a defector again within a short time. During the lifetime of the new cooperator, however, the other neighboring defector adopts the C strategy very probably from the cooperator in the original triangle (if $b < 3/2$), and they will form a neighboring (stable) triplet of cooperators. The iteration of these processes yields a growing tree of cooperator triplets. The growing process is stopped at the sites that separate two branches of the growing tree, because the corresponding defector(s) can exploit two or more cooperators simultaneously. The blocking events occur frequently for the spatial structures and are excluded for treelike structures. Thus, the absence of loops (formed by the one-site overlapping triangles) sustains the spreading of cooperation in RRG2. This picture is valid if only one tree of cooperator triplets exists in the initial state. For a random initial state, however, the system will have many (trees of) cooperator triplets which will check each other in the growing process at the sites where their branches are separated even on the RRG2 structure. In the final stationary state the spreading and blocking effects are balanced in a way that takes the noise into account. Evidently, the noisy effects can break up the triplets of cooperators. As a result, one can observe monotonic decrease in the function $\rho(K)$ when increasing K for any $b < b_{cr}$.

Disregarding the triangles, the presence of loops in the connectivity structure reduces the measure of cooperation for high noise level too. Figure 5 clearly demonstrates that RRG2 is more advantageous than the Kagomé lattice in the maintenance of cooperation. Similar conclusion can be deduced when we compare the results obtained on the RRG1 and square lattices (see Fig. 5). In these cases the stable

triplets of cooperators cannot be observed because of the absence of triangles in the connectivity structure. For both structures the maintenance of cooperation is supported by noisy events resembling coherence resonance [29]. At finite noise level the occasional (irrational) adoption of the neighboring C strategy increases the tree of cooperators, and this effect is weakened by the presence of loops as described above.

It is more surprising that the disadvantageous presence of loops is so relevant that RRG1 becomes the most efficient structure to sustain the cooperation if K exceeds a threshold value ($K > K_{th} \approx 0.4$) for $z=4$. In other words, in the maintenance of cooperation the advantageous effect of the quenched (regular) neighborhood is weakened by the spatial characteristics of the connectivity structure.

To summarize, we have systematically studied the effect of noise K and temptation b to choose defection on the measure of cooperation in an evolutionary PD game for two types of random regular connectivity structures with $z=4$.

For this purpose we have determined the critical value $b_{cr}(K)$ of temptation until the cooperators can remain alive. For sufficiently high noise levels the comparison of different connectivity structures indicates that the highest $b_{cr}(K)$ can be achieved by minimizing the number of loops in the connectivity structure. On the contrary, at low noise levels, the preferred structure is built up randomly from overlapping triangles in a way that the overlapping triangles have only one common site. Evidently, the analysis becomes more complicated for those connectivity structures which involve variation in degree, $z > 4$, and more complex connections at the overlapping triangles.

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